J. Fluid Mech. (1973), vol. 58, part 4, pp. 703–708
Printed in Great Britain

Wave velocity in parallel flows of a viscous fluid

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(Received 8 December 1972)

It is shown in this note that the velocity of unstable or neutral waves in plane Poiseuille flow or plane Couette–Poiseuille flow, or of axisymmetric waves in Poiseuille flow, stable or unstable, must lie within the range of the velocity of the flow.

1. Introduction

Bounds for the real part of the complex wave velocity c appearing in the Orr–Sommerfeld equation have been given by Synge (1938) for plane Couette and Poiseuille flows. Synge's results were extended by Pai (1954) to apply to more general flows, and Pai's results were sharpened by Joseph (1968). For the plane Couette flow it is already evident from Synge's results that the real part of $c = c_r + ic_i$ must be within the range of the velocity U of the flow, whatever its imaginary part may be. For other flows the bounds of c_r given by all three authors mark an interval greater than the range of U; but their results are for all possible values of c_i, hence for all kinds of waves – amplified, neutral or damped.

In this note we shall derive some results from which we deduce immediately that the velocity c_r of unstable and neutral waves (in fact, even some damped ones) in plane Poiseuille and plane Couette–Poiseuille flows (combinations of a plane Couette flow with a plane Poiseuille flow) must be within the range of the velocity of the flow. Furthermore, we shall show that axisymmetric disturbances in Poiseuille flow (in circular pipes) always propagate with a velocity which is within the range of the velocity of the flow, whether the disturbances are stable, neutral or unstable.

2. Recapitulation of some known results for plane flows

The Orr–Sommerfeld equation is

$$(D^2-\alpha^2)^2\phi = i\alpha R[(U-c)(D^2-\alpha^2)\phi - U''\phi], \tag{1}$$

in which the unknown ϕ is the (complex) amplitude of the stream function, α is the wavenumber, R the Reynolds number, U the dimensionless velocity (in the x direction), c the complex wave velocity $c_r + ic_i$ and

$$D = d/dy, \quad U'' = D^2U,$$

y being the dimensionless Cartesian co-ordinate normal to the direction of flow, on which alone U depends. Since the stream function is

$$\psi = \phi(y)\exp\{i\alpha(x-ct)\},$$

the rate of growth of the waves is αc_i, and the waves are stable, neutral or unstable according as c_i is negative, zero or positive. The parameters α and R are by definition positive. With ϕ' denoting $D\phi$, the boundary conditions are

$$\phi(0) = \phi'(0) = 0 = \phi(1) = \phi'(1). \tag{2}$$

By multiplying (1) by ϕ^*, the complex conjugate of ϕ, and integrating, using (2) whenever necessary, Synge (1938) obtained

$$\int |(D^2-\alpha^2)\phi|^2 + \alpha R c_i \int P = i\alpha R\left[-\int (U-c)P + \int U'\phi\phi'^*\right], \tag{3}$$

in which

$$P = |\phi'|^2 + \alpha^2|\phi|^2,$$

the limits of integration are understood to be zero and one, and the differential dy is omitted for all integrals, for convenience. Since

$$\mathrm{Re}\int U'\phi\phi'^* = -\tfrac{1}{2}\int U''|\phi|^2,$$

we conclude from (3), upon taking its imaginary part, that

$$\int (U-c_r)P + \tfrac{1}{2}\int U''|\phi|^2 = 0. \tag{4}$$

From (4) we obtain the following known results which were already evident from the results of Synge [see equation (3) of Joseph (1968), which is Synge's result]:

$$c_r > U_{\min} \quad \text{if} \quad U'' \geqslant 0 \text{ throughout}, \tag{5}$$

$$c_r < U_{\max} \quad \text{if} \quad U'' \leqslant 0 \text{ throughout}, \tag{6}$$

$$U_{\min} < c_r < U_{\max} \quad \text{if} \quad U'' = 0 \text{ throughout}. \tag{7}$$

These results of Synge are for any waves (or disturbances), amplified, neutral or damped.

3. New results for unstable and neutral waves in plane flows

Let $F(y)$ be defined by

$$(U-c)F = \phi. \tag{8}$$

Note that F is never infinite if $c_i \neq 0$, or if c_r is *outside* the range of U. The Orr–Sommerfeld equation then becomes

$$(D^2-\alpha^2)^2[(U-c)F] = i\alpha R\{[(U-c)^2F']' - \alpha^2(U-c)^2F\}, \tag{9}$$

in which the prime indicates differentiation with respect to y. The boundary conditions on F are, so long as $U-c$ does not vanish on the boundary (we shall return to this point later),

$$F(0) = F'(0) = 0 = F(1) = F'(1). \tag{10}$$

Multiplying (9) by F^*, the complex conjugate of F, and integrating between zero and one, using (10) whenever necessary, we obtain

$$\begin{aligned}\int (U-c_r)\,|(D^2-\alpha^2)F|^2 - 2\int U''|F'|^2 + \tfrac{1}{2}\int U^{\mathrm{iv}}|F|^2 - ic_i\int |(D^2-\alpha^2)F|^2 - \alpha^2\int U''|F|^2 \\ + \int U'(F'F^{*''} - F^{*'}F'') - \tfrac{1}{2}\int U'''(FF^{*'} - F'F^*) + \alpha^2\int U'(FF^{*'} - F^*F') \\ = i\alpha R\{-\int[(U-c_r)^2 - c_i^2)Q + i2c_i\int (U-c_r)Q\},\end{aligned} \tag{11}$$

in which $$Q = |F'|^2+\alpha^2|F|^2. \tag{11a}$$

The last four terms on the left-hand side of (11) are imaginary. Taking the real part of (11), we obtain

$$\int(U-c_r)\,[|(D^2-\alpha^2)\,F|^2+2\alpha Rc_i\,Q]-2\int U''|F'|^2+\tfrac{1}{2}\int U^{\mathrm{iv}}|F|^2-\alpha^2\int U''|F|^2 = 0. \tag{12}$$

From this equation we conclude that

$$U_{\max} > c_r \quad \text{if} \quad c_i > 0, \quad \text{and} \quad U'' \geqslant 0, \quad U^{\mathrm{iv}} \leqslant 0 \text{ throughout}, \tag{13}$$

$$U_{\min} < c_r \quad \text{if} \quad c_i > 0, \quad \text{and} \quad U'' \leqslant 0, \quad U^{\mathrm{iv}} \geqslant 0 \text{ throughout}. \tag{14}$$

If $c_i = 0$, the function F is well defined if $c_r > U_{\max}$ or $c_r < U_{\min}$, and (12) states that, for neutral waves,

$$c_r > U_{\max} \text{ is impossible, or } U_{\max} \geqslant c_r, \text{ if } U'' \geqslant 0,\ U^{\mathrm{iv}} \leqslant 0 \text{ throughout}, \tag{15}$$

$$c_r < U_{\min} \text{ is impossible, or } U_{\min} \leqslant c_r, \text{ if } U'' \leqslant 0,\ U^{\mathrm{iv}} \geqslant 0 \text{ throughout}. \tag{16}$$

For plane Poiseuille flow or plane Couette–Poiseuille flows, U'' is either positive throughout or negative throughout, and U^{iv} is zero. Hence for such flows (5), (6), (13) and (14) give

$$U_{\min} < c_r < U_{\max} \quad \text{for} \quad c_i > 0, \tag{17}$$

and (5), (6), (15) and (16) give

$$U_{\min} < c_r \leqslant U_{\max} \quad \text{if} \quad U'' > 0, \quad c_i = 0, \tag{18}$$

$$U_{\min} \leqslant c_r < U_{\max} \quad \text{if} \quad U'' < 0, \quad c_i = 0. \tag{19}$$

In the next section we shall show that for neutral waves in plane Poiseuille or plane Couette–Poiseuille flows c_r cannot attain $U_{\max}$ or $U_{\min}$. This point is important whenever one attempts to calculate the neutral-stability curves by asymptotic methods, for if $c_r = U_{\max}$ or $c_r = U_{\min}$ the critical layer would always include the boundary, however large the Reynolds number, and the existing calculations would break down.

4. Neutral and some damped modes in plane flows

We now concentrate on plane Poiseuille and plane Couette–Poiseuille flows. Thus

$$U'' = \text{constant}, \quad U^{\mathrm{iv}} = 0.$$

We shall define a new F by

$$(U-c_1)\,F = \phi, \tag{20}$$

where $$c_1 = c_r+ic_{1i} = c_r+i[c_i+4\pi^2(\alpha R)^{-1}]. \tag{21}$$

Note that F is never infinite if

$$c_i > -4\pi^2(\alpha R)^{-1}.$$

The Orr–Sommerfeld equation can then be written as

$$(D^2-\alpha^2)^2\,\phi+4\pi^2(\phi''-\alpha^2\phi) = i\alpha R[(U-c_1)\,(\phi''-\alpha^2\phi)-U''\phi],$$

or

$$(D^2-\alpha^2)\,[(D^2-\alpha^2)+4\pi^2]\,[(U-c_1)\,F] = i\alpha R\{[(U-c_1)^2\,F']'-\alpha^2(U-c_1)^2\,F\}, \tag{22}$$

with the boundary conditions

$$F(0) = F'(0) = 0 = F(1) = F'(1). \tag{23}$$

By multiplying (22) by F^* and integrating between zero and 1, using (23) whenever necessary, we obtain

$$\begin{aligned} T - i4\pi^2(\alpha R)^{-1} \int |(D^2-\alpha^2)F|^2 - 4\pi^2 \int (U-c_1)Q - 4\pi^2 \int U'FF'^* \\ = -i\alpha R \int (U-c_1)^2 Q = -i\alpha R\left\{\int [(U-c_r)^2 - c_{1i}^2]Q - 2ic_{1i}\int (U-c_r)Q\right\}, \end{aligned} \tag{24}$$

in which Q is defined by (11 a), with the F therein defined by (20), and T stands for terms on the left-hand side of (11), with the F now defined by (20). Noting that

$$\operatorname{Re} \int U'FF'^* = -\tfrac{1}{2}\int U''|F|^2,$$

and taking the real part of (24), we have (now that U'' is constant and $U^{\text{iv}} = 0$)

$$\int (U-c_r)\left[|(D^2-\alpha^2)F|^2 + 2(\alpha R c_i + 2\pi^2)Q\right] = 2U''I + \alpha^2 \int U''|F|^2, \tag{25}$$

in which
$$I = \int (|F'|^2 - \pi^2|F|^2) \geqslant 0, \tag{26}$$

as is well known. Thus (25) gives

$$c_r < U_{\max} \quad \text{if} \quad U'' > 0, \quad c_i \geqslant -2\pi^2(\alpha R)^{-1}, \tag{27}$$

$$U_{\min} < c_r \quad \text{if} \quad U'' < 0, \quad c_i \geqslant -2\pi^2(\alpha R)^{-1}. \tag{28}$$

Note that for
$$c_i \geqslant -2\pi^2(\alpha R)^{-1} \tag{29}$$

F is certainly well defined (i.e. it never becomes infinite) by (20), for any values of c_r. On combining (27) and (28) with (5) and (6), we conclude that when (29) is satisfied

$$U_{\min} < c_r < U_{\max}$$

for plane Poiseuille and plane Couette–Poiseuille flows; or, ignoring the damped modes, we may state the results in the following theorem

THEOREM 1. The velocity (c_r) of neutral or unstable shear waves in plane Poiseuille flow or in plane Couette–Poiseuille flows must be within the range of the velocity of the flow, and the maximum or minimum of the flow velocity is never attained by c_r.

Recalling also that c_i is bounded (Yih 1969) above by

$$h = \frac{q}{2\alpha} - \frac{\lambda^2}{\alpha R} \tag{30}$$

with
$$\lambda^2 = \min \frac{I_2 + 2\alpha^2 I_1 + \alpha^4 I_0}{I_1 + \alpha^2 I_0},$$

$$I_2 = \int |\phi''|^2, \quad I_1 = \int |\phi'|^2, \quad I_0 = \int |\phi|^2,$$

$$q = \max |U'(y)|,$$

we can state the following theorem.

THEOREM 2. For plane Poiseuille or plane Couette–Poiseuille flows, the eigenvalues c for neutral and unstable waves lie inside or on the horizontal boundaries of a rectangle in the complex-c plane which has the range of the flow velocity as its base and h defined by (30) as its height, provided h is positive.†

This theorem (a rectangle theorem) corresponds to Howard's semicircle theorem for the Rayleigh equation or the differential equation for flows of an inviscid stratified fluid (Howard 1961), and I present it to Professor Howard in return for the delight that his semicircle theorem has given me.

5. Results for Poiseuille flow

For Poiseuille flow the mean velocity is given by

$$W(r) = W_0(1-r^2),$$

in which r is the radial distance from the centre-line of the circular pipe, measured in units of the pipe radius r_0. For axisymmetric disturbances the Stokes stream function ψ can be used, and the perturbation part of it, denoted by ψ', has the form

$$\psi' = \phi(r)\exp\{i\alpha(z-ct)\},$$

in which z is measured along the centre-line in units of r_0, α is the dimensionless wavenumber, c is the complex wave velocity (c_r+ic_i) measured in units of W_0, and t is the time, measured in units of r_0/W_0. Then the equation governing stability is‡

$$(L-\alpha^2)^2\phi = i\alpha R(1-r^2-c)(L-\alpha^2)\phi, \tag{31}$$

in which $R = W_0 r_0/\nu$ is the Reynolds number and

$$L = \frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2} = \frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right) = \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\ )\right].$$

The boundary conditions are (if primes denote d/dr)

$$\phi(1) = 0 = \phi'(1), \quad \phi(r) \quad \text{regular at} \quad r = 0. \tag{32}$$

If (31) is multiplied by $r\phi^*\,dr$ and integrated from zero to one, the result is, after (32) has been applied,

$$I_2+2\alpha^2I_1+\alpha^4I_0 = -i\alpha R\int_0^1(1-r^2-c)\,Q\,dr+i2\alpha R\int_0^1(r\phi)'(r\phi^*)\,dr, \tag{33}$$

in which
$$Q = r^{-1}|(r\phi)'|^2+\alpha^2r|\phi|^2,$$

$$I_0\int_0^1 r|\phi|^2\,dr, \quad I_1 = \int_0^1\frac{1}{r}|(r\phi)'|^2\,dr, \quad I_2 = \int_0^1 r|L\phi|^2\,dr.$$

Since
$$J \equiv \int_0^1(r\phi)'(r\phi^*)\,dr = -\int_0^1(r\phi^*)'(r\phi)\,dr,$$

the real part of J is zero. Taking the imaginary part of (33), we have

$$\int_0^1(1-r^2-c_r)\,Q\,dr = 0,$$

† If h is zero or negative for some R, the flow is stable (at most neutrally stable) for that value of R, and we do not need theorem 2.

‡ Apart from slight changes in notation, this is equation (1.3.34) in Lin (1955, p. 10), in which σ should read σR.

which demands that $$0 < c_r < 1, \tag{34}$$
which states that c_r is within the range of W/W_0, or that the dimensional wave velocity $c_r W_0$ is within the range of W.

We shall now establish an upper bound for c_i. First we note that the regularity of $\phi(r)$ at $r = 0$ implies that $\phi(0) = 0$, as indeed it must, for otherwise the radial part (u) of the perturbation velocity would be infinite there. Since

$$\phi(1) = 0 = \phi(0) \quad \text{and} \quad 0 \leqslant r \leqslant 1,$$

we have $$I_1 \equiv \int (1/r)\,|(r\phi)'|^2 > \int |(r\phi)'|^2 \geqslant \pi^2 \int |r\phi|^2, \tag{35}$$
as is well known. Furthermore, Schwarz's inequality gives

$$i2 \int (r\phi)'\,(r\phi^*) \leqslant 2 \int |(r\phi)'|\,|r\phi^*| \leqslant \int |(r\phi)'|^2 + \int |r\phi|^2. \tag{36}$$

Hence $$i2J \leqslant \frac{\pi^2+1}{\pi^2}\int |(r\phi)'|^2 < \frac{\pi^2+1}{\pi^2} I_4, \quad \text{with} \quad I_4 = \int Q\,dr. \tag{37}$$

Taking the real part of (33), we have

$$-\alpha R c_i I_4 + i2\alpha R J = I_2 + 2\alpha^2 I_1 + \alpha^4 I_0,$$

from which we deduce, using (37), the result

$$c_i < h \equiv < \frac{\pi^2+1}{\pi^2} - \frac{\lambda^2}{\alpha R}, \tag{38}$$

where $$\lambda^2 = \min \frac{I_2 + 2\alpha^2 I_1 + \alpha^4 I_0}{I_4}. \tag{39}$$

The value of λ^2 can be calculated from (39) and the boundary conditions on ϕ. We shall state the results obtained in this section in the following two theorems.

THEOREM 3. For Poiseuille flow, the velocity c_r of all shear waves, whether they be stable, neutral or unstable, must be within the range of the velocity of the flow, the maximum or minimum of which is never attained by c_r.

THEOREM 4. For Poiseuille flow, the eigenvalues c for neutral or unstable waves lie inside or on the lower horizontal boundary of a rectangle in the complex-c plane, which has the range of the flow velocity as its base and h defined in (38) as its height, provided h is positive.

This work has been jointly supported by the National Science Foundation and the Office of Naval Research.

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